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AN ANALYSIS ON ZERO-DIVISOR GRAPH OF A RING

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Abstract. Throughout this paper we analyze the ideal zero divisor graph centred on $\Gamma_1(\mathcal{R})$ of the commutative ring $\mathcal{R}$. We address some of the theoretical graph properties of $\Gamma_1(\mathcal{R})$ in relation to the zero divisor graph by characterizing the condition in which clique number and chromatic number is same..

Keywords. zero-divisor graph, chromatic number.

1. *Introduction*

An interesting topic for mathematicians is the study of algebraical graphics theory which dates back to 1973 at least when n. Biggs introduced his own work. The main objective of Bigg's research was to transcribe the characteristics of graphs into algebraic structures, and using the findings and methodology of algebra, to conclude the graph theorems. Anderson and Naseer extended this analysis of coloration of the zero-divisor graph on a commuting ring [4]. In the meantime the graph of zero divisors haas mainly focused the attention of the algebraist and graph theorist. We provide the same goal as N. Bigg [5] in this research paper. Having 1 as identity and $Z(\mathcal{R})$ be its zero divisors of a commuting Ring . Assuming $\Gamma(\mathcal{R})$ is an undirected graph with non-zero divisors of $\mathcal{R}$ with 2 different cartesian coordinates x and y connected by edge only when $xy = 0$. DeMeyer [3] et al. and Redmond [8] generalized the graph with zero divisor to other algebraic structures.
Assuming $\mathcal{R}$as a commuting ring and $\mathcal{I}$ as R′s ideal. The optimal based graph is an undirected graph Γ_1(R)having vertex $\{x \in \mathcal{R} - \mathcal{I} : xy \in I$ for some $y \in \mathcal{R} - \mathcal{I}\}$ with corresponding adjacent vertices x and y, if and only if $xy \in I$. The ideal graph is an undirected graph. Redmond [9] incorporated it. In [9], he observed in reference to the graph of the zero splitter

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values parameters as connectivity, clique, diameter, girth, etc. Various other study activities are still taking place here. The values of specifications such as vertex chromatic number, number of cliques, the maximum and minimum degree etc. are found in this paper. Section 2 provides the description and theorem for the following sections from [9]. For a graph G, the degree deg(v) of a vertex v in G is the number of edges incident with v. deg(v) represents the degree of the vertex v in $\Gamma_1(\mathcal{R})$.The minimum and maximum vertical grades of G are represented by $\delta(G)$ and $\Delta(G)$, respectively. If all vertices of G are the same, a map G is regular. If any vertex of G is of degree 1 a graph G is one-factor. The complete diagram, with n vertices and complete bipartite diagram, is indicated by two sections m and n by K_n and $K_{n,m}$. Unless the inducted subgraph on $\varphi$is a complete graph, a subset φ of the vertices of G is considered a clique. The vertices in set φ are labelled with $|\varphi|$.

2. Preliminaries

Definition 2.1 [9]. Let $\mathcal{R}$be a commutative ring and let I be an ideal of $\mathcal{R}$. The ideal based zero divisor graph is an undirected graph $\Gamma_1(\mathcal{R})$ with vertices $\{x \in \mathcal{R} - \mathcal{I} : xy \in \mathcal{I}$for some $y \in \mathcal{R} - \mathcal{I}\}$, where distinct vertices x and y are adjacent if and only if $xy \in \mathcal{I}$.

Definition 2.2 [9]. Using the notation as in the above construction, we call the subset $a_\lambda + \mathcal{I}$ a column of $\Gamma_1(\mathcal{R})$. If $a^2 \in \mathcal{I}$, then we call $a_\lambda + \mathcal{I}$ a connected column of $\Gamma_1(\mathcal{R})$.

Theorem 2.3 [7, Theorem 7]. For a finite ring $\mathcal{R}$, if $\Gamma(\mathcal{R})$ is a regular graph, then it is either a complete graph or a complete bipartite graph.

Theorem 2.4 [9, Theorem 5.7]. Let$\mathcal{I}$ be a nonzero ideal of a ring $\mathcal{R}$. Then $\Gamma_1(\mathcal{R})$ is bipartite if and only if either

(a) $gr\left(\Gamma_1(\mathcal{R})\right) = \infty$ or

(b) $gr(\Gamma_1(\mathcal{R})) = 4$ and $\Gamma(\frac{\mathcal{R}}{\mathcal{I}})$ is bipartite.

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Theorem 2.5 [2, Theorem 2.8]. Let $\mathcal{R}$ be a commutative ring. Then $\Gamma'(\mathcal{R})$ is a complete graph if and only if either $\tilde{\mathcal{R}} = Z_2 \times Z_2$ or $xy = 0$ for every $x, y \in Z(\mathcal{R})$. In particular, if $\mathcal{R}$ is a reduced commutative ring and not a field, then $\Gamma'(\mathcal{R})$ is a complete graph if and only if $\tilde{\mathcal{R}} \cong Z_2 \times Z_2$

3. *Main Results*

In this section we characterize when the chromatic number and clique number of $\Gamma'_1(\mathcal{R})$ are equal and we also prove the following relationship

$$\mu\left(\Gamma'\left(\frac{\mathcal{R}}{\mathcal{I}}\right)\right) \leq |\mathcal{I}|\mu\left(\Gamma'\left(\frac{\mathcal{R}}{\mathcal{I}}\right)\right)$$

and

$$\wp(\Gamma'_1(\mathcal{R})) \leq |\mathcal{I}|\ \wp(\Gamma'(\frac{\mathcal{R}}{\mathcal{I}})).$$

Theorem 3.1. Let $\mathcal{R}$ be a commutative ring and $\mathcal{I}$ be an ideal of $\mathcal{R}$. If

$$\wp\left(\Gamma'\left(\frac{\mathcal{R}}{\mathcal{I}}\right)\right) = \mu\left(\Gamma'\left(\frac{\mathcal{R}}{\mathcal{I}}\right)\right),$$

Then

$$\wp(\Gamma'_1(\mathcal{R}))) = \mu(\Gamma'_1(\mathcal{R})).$$

Proof. Considering,

$$\wp(\Gamma'(\frac{\mathcal{R}}{\mathcal{I}})) = \omega(\Gamma'(\frac{\mathcal{R}}{\mathcal{I}})) = k.$$

Assuming

$$V(\Gamma'(\frac{\mathcal{R}}{\mathcal{I}})) = \{a_i + \mathcal{I} : i \in \Lambda\}.$$

Assuming, different color classes as $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_l$ of $\Gamma'\left(\frac{\mathcal{R}}{\mathcal{I}}\right)$. It is given that,

$$\omega(\Gamma'(\frac{\mathcal{R}}{\mathcal{I}})) = l,$$

There is an existence of elements $\alpha_1 + \mathcal{I}, \alpha_2 + \mathcal{I}, \alpha_3 + \mathcal{I}, \ldots, \alpha_l + \mathcal{I}$ in such a manner thatno two of them are both in the same colour class. With no generality failure, let $\alpha_i + \mathcal{I} \in \mathcal{A}_i$ for each and every i. Assuming

$$\mathcal{S} = \{\alpha_1 + \mathcal{I}, \alpha_2 + \mathcal{I} \ldots, \alpha_l + \mathcal{I}\}.$$

Then $< S >$ is a subgraph of $\Gamma(\frac{\mathcal{R}}{\mathcal{I}})$.

Let $\mathcal{M} = \{\alpha_l : \alpha_l + \mathcal{I} \in \mathcal{S}\} \cup \{\alpha_l + a : \alpha_l + \mathcal{I} \in \mathcal{S}, \alpha_2 \in \mathcal{I}, a \in \mathcal{I}^*\}$
which is maximal as well as complete.

It is given that $< S >$is maximal and hence $< M >$ is a maximal as well as complete subgraph in $\Gamma(\frac{\mathcal{R}}{\mathcal{I}})$, which gives $\mu(\Gamma_1(\mathcal{R})) \geq |\mathcal{M}|$.It implies that the colorvertices of $\mathcal{M}$ with $|\mathcal{M}|$ are all different. Obviously$\alpha + \mathcal{I}$ an independent set for α^2doesn't belongs to $\mathcal{I}$ with $\alpha + \mathcal{I} \in \mathcal{S}$ is inducedin $\Gamma_1(\mathcal{R})$ and so color that the vertices $\alpha + a$ according to the color of α, for all $a \in \mathcal{I}^*$.

Now, assuming$\varphi = \{\alpha : \alpha + \mathcal{I} \in \mathcal{S}\}$ resulting φas having distinct colors. For each $\mathcal{Y}$ doesn't belongs to φ, $\mathcal{Y} = \alpha_z + a$, where $a \in \mathcal{I}$ and z doesn't belongs to $\mathbb{N}_k = \{n : \mathbb{N} \cap \{1,2,3,4,5 \ldots, k\}\}$where $\mathbb{N}$ represents natural number system and $\mathcal{Y} + \mathcal{I} = \alpha_z + \mathcal{I}$.It is given that $\alpha_z + \mathcal{I} \in \mathcal{A}_i$ also the$\mathcal{A}_i$'s are not dependent, the color of the vertices $\alpha_z + a$ with the color of $\alpha_h + a$. It can easily be concluded that the vertices which doesn't belongs to φ in the similar manner it can easily be concluded that the coloring is proper. Hence

$$\wp(\Gamma_1(\mathcal{R})) \leq |\mathcal{M}|.$$

Because,

$$\mu(\Gamma_1(\mathcal{R})) \leq \wp(\Gamma_1(\mathcal{R})), \qquad \wp(\Gamma_1(\mathcal{R})) = \mu(\Gamma_1(\mathcal{R})).$$

Theorem 3.2. Assuming $\mathcal{R}$as a commutative ring and $\mathcal{I}$ as an ideal of $\mathcal{R}$. Then
$\wp(\Gamma_1(\mathcal{R})) \leq |\mathcal{I}| \wp(\Gamma(\mathcal{R}))$.

Proof. Considering

$$\wp(\Gamma(\mathcal{R})) = l.$$

Assuming arbitrarily chosen$\alpha_1 + \mathcal{I}, \alpha_2 + \mathcal{I} \ldots, \alpha_l + \mathcal{I}$in $\Gamma(\mathcal{R})$in such a manner maximal complete subgraph is induced by

$$\mathcal{B} = \bigcup_{1 \leq i \leq l} \{\alpha_i + \mathcal{I}\}.$$

Consider

$$\mathcal{S} = \{\alpha + h : \alpha + \mathcal{J} \in \mathcal{B}, h \in \mathcal{J}\}.$$

Assuming,$\alpha_2 \in \mathcal{J}$ for all $\alpha + \mathcal{J} \in \mathcal{B}$.Since in $\Gamma_1(\mathcal{R})$we have $< S >$ is a complete subgraph. If we havea complete subgraph in $\Gamma_1(\mathcal{R})$ namely $\mathcal{S} \cup \{j\}$ is, then $j(\alpha + h)$belongs to $\mathcal{J}$ and so $(j + \mathcal{J})(\alpha + \mathcal{J}) = 0 + \mathcal{J}$. So a clique of size $l + 1$ is generated by $\mathcal{B} \cup \{j + \mathcal{J}\}$, which itself is a contradictory statement. And hence$< S >$is maximal and and also

$$\wp(\Gamma_1(\mathcal{R})) == |\mathcal{S}|.$$

Results,

$$\wp(\Gamma_1(\mathcal{R})) = |\mathcal{J}| \wp(\Gamma(\mathcal{R})).$$

But in all 0ther instances,

$$\wp(\Gamma_1(\mathcal{R})) < |\mathcal{J}| \wp(\Gamma(\mathcal{R}))$$

and so that's the result.

Theorem 3.3. Assuminga nonempty ideal$\mathcal{J}$ of $\mathcal{R}$ (commutative ring). If it is considered that$\Gamma_1(\mathcal{R})$ is a graph on a single vertex,

$$\wp(\Gamma_1(\mathcal{R})) = |\mathcal{J}|.$$

Proof. Considering$\Gamma_1(\mathcal{R})$ having only one vertex, then $\Gamma_1(\mathcal{R})$ comes from a single connected column and thus is the complete graph $|\mathcal{J}|$, which is followed by the result.

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